1. Retailers and Wholesalers

Retailers

Consider the final-good aggregator

$$Y_t = \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}}\, di\right]^{\frac{\theta}{\theta-1}}.$$

The problem of retailers is:

$$\min_{\{Y_t(i)\}} \int_0^1 P_t(i) Y_t(i)\, di \quad \text{s.t.} \quad Y_t = \left[\int_0^1 Y_t(i)^{\frac{\theta-1}{\theta}}\, di\right]^{\frac{\theta}{\theta-1}}.$$

(i) Set up the Lagrangian and derive the FOC with respect to $Y_t(i)$.

(ii) Given the price index

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta}\, di\right]^{\frac{1}{1-\theta}}.$$

derive the demand schedule for variety i:

$$Y_t(i) = Y_t \left(\frac{P_t(i)}{P_t}\right)^{-\theta}.$$

(iii) Show that

$$\frac{d \log Y_t(i)}{d \log P_t(i)} = -\theta.$$

(iv) Define

$$\mu \equiv \frac{1}{\theta - 1}.$$

Explain why the markup is decreasing in θ.

Wholesalers

Each wholesaler produces according to

$$Y_t(i) = A_t N_t(i),$$

and faces demand

$$Y_t(i) = Y_t \left(\frac{P_t(i)}{P_t}\right)^{-\theta}.$$

Its problem is

$$\max_{P_t(i), N_t(i)} (1-\tau_t)\frac{P_t(i)}{P_t} Y_t(i) - \frac{W_t}{P_t} N_t(i)$$

subject to

$$Y_t(i) = A_t N_t(i), \qquad Y_t(i) = Y_t \left(\frac{P_t(i)}{P_t}\right)^{-\theta}.$$

Derive real marginal cost:

$$MC_t = \frac{W_t}{A_t P_t}.$$

(i) Rewrite the firm's problem as a problem in $P_t(i)$ only.

(ii) Derive the FOC and show that

$$\frac{P_t(i)}{P_t} = MC_t \frac{\theta}{(1-\tau_t)(\theta-1)}.$$

(iii) Using $\mu = \frac{1}{\theta-1}$, rewrite the pricing rule as

$$\frac{P_t(i)}{P_t} = MC_t \frac{1+\mu}{1-\tau_t}.$$

(iv) Explain why monopolistic competition alone is not enough to make monetary policy non-neutral.

2. Sticky prices

Assume Calvo pricing:

- with probability $1-\alpha$, a firm resets its price at time t,
- with probability α, it keeps the previous price.

Calvo problem

A firm resetting at time t chooses $\widetilde{P}_t(i)$ to solve

$$\max_{\widetilde{P}_t(i)} E_t \sum_{j=0}^{\infty} \alpha^j M_{t,t+j} \left[(1-\tau_{t+j}) \frac{\widetilde{P}_t(i)}{P_{t+j}} - MC_{t+j} \right] Y_{t+j|t}(i)$$

subject to

$$Y_{t+j|t}(i) = Y_{t+j} \left(\frac{\widetilde{P}_t(i)}{P_{t+j}} \right)^{-\theta},$$

where

$$M_{t,t+j} \equiv \beta^j \left(\frac{C_{t+j}}{C_t} \right)^{-\sigma}.$$

(i) Substitute the demand schedule into the objective function.

(ii) Derive the FOC with respect to $\widetilde{P}_t(i)$.

(iii) Define

$$\Pi_{t+j,t} \equiv \frac{P_{t+j}}{P_t}.$$

Show that the reset-price condition can be written as

$$\frac{\widetilde{P}_t(i)}{P_t} = (1+\mu) \frac{E_t \sum_{j=0}^{\infty} \alpha^j M_{t,t+j} \Pi^{\theta}_{t+j,t} MC_{t+j} Y_{t+j}}{E_t \sum_{j=0}^{\infty} \alpha^j M_{t,t+j} (1-\tau_{t+j}) \Pi^{\theta-1}_{t+j,t} Y_{t+j}}.$$

(iv) Explain the economic interpretation of the numerator and denominator.

(v) In symmetric equilibrium, all adjusting firms choose the same reset price. Write

$$\frac{\widetilde{P}_t}{P_t} = \frac{X_{1t}}{X_{2t}}$$

and define X_{1t} and X_{2t}.

(vi) Show that we can rewrite X_{1t} and X_{2t} as:

$$X_{1t} = (1+\mu)MC_tY_t + \alpha\beta E_t\left[\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\Pi_{t+1}^{\theta}X_{1,t+1}\right],$$

$$X_{2t} = (1-\tau_t)Y_t + \alpha\beta E_t\left[\left(\frac{Y_{t+1}}{Y_t}\right)^{-\sigma}\Pi_{t+1}^{\theta-1}X_{2,t+1}\right],$$

Aggregate price level

The CES price index is

$$P_t^{1-\theta} = \int_0^1 P_t(i)^{1-\theta}\,di.$$

(i) Using the Calvo structure, show that

$$P_t^{1-\theta} = \alpha P_{t-1}^{1-\theta} + (1-\alpha)\widetilde{P}_t^{1-\theta}.$$

(ii) Using the result from the previous section, show that

$$1 = \alpha\Pi_t^{\theta-1} + (1-\alpha)\left(\frac{X_{1t}}{X_{2t}}\right)^{1-\theta}.$$

Price dispersion

Define

$$\Delta_t \equiv \int_0^1 \left(\frac{P_t(i)}{P_t}\right)^{-\theta} di.$$

(i) Starting from $Y_t(i) = A_tN_t(i)$ and the demand schedule, show that aggregate production satisfies

$$Y_t\Delta_t = A_tN_t.$$

(ii) Explain why $\Delta_t > 1$ represents a resource misallocation.

(iii) Split firms into adjusters and non-adjusters and derive

$$\Delta_t = \alpha\Pi_t^{\theta}\Delta_{t-1} + (1-\alpha)\left(\frac{\widetilde{P}_t}{P_t}\right)^{-\theta}.$$

(iv) Use the price-index relation to show that

$$\left(\frac{\widetilde{P}_t}{P_t}\right)^{1-\theta} = \frac{1-\alpha\Pi_t^{\theta-1}}{1-\alpha}.$$

(v) Deduce the law of motion

$$\Delta_t = \alpha\Pi_t^{\theta}\Delta_{t-1} + (1-\alpha)\left(\frac{1-\alpha\Pi_t^{\theta-1}}{1-\alpha}\right)^{\frac{\theta}{\theta-1}}.$$

3. Households

The representative household solves

$$\max_{\{C_t,B_t,N_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\varphi}}{1+\varphi} \right)$$

subject to

$$P_t C_t + B_t = (1+i_{t-1})B_{t-1} + W_t N_t.$$

(i) Derive the Euler equation:

$$C_t^{-\sigma} = \beta E_t \left[C_{t+1}^{-\sigma} \frac{1+i_t}{\Pi_{t+1}} \right].$$

(ii) Derive the labour-supply condition:

$$\frac{W_t}{P_t} = N_t^{\varphi} C_t^{\sigma}.$$

(iii) Use the goods market clearing condition and the equation for the marginal cost to show that

$$A_t MC_t = N_t^{\varphi} C_t^{\sigma}.$$

(iv) Using a result from section 1, derive

$$A_t^{1+\varphi} MC_t = Y_t^{\sigma+\varphi} \Delta_t^{\varphi}.$$

4. Nonlinear equilibrium

Given exogenous productivity A_t and policy rate i_t, equilibrium is a sequence

$$\{Y_t, MC_t, \Pi_t, \Delta_t, X_{1t}, X_{2t}\}$$

satisfying

$$Y_t^{-\sigma} = \beta E_t \left[Y_{t+1}^{-\sigma} \frac{1+i_t}{\Pi_{t+1}} \right],$$

$$A_t^{1+\varphi} MC_t = Y_t^{\sigma+\varphi} \Delta_t^{\varphi},$$

$$1 = \alpha \Pi_t^{\theta-1} + (1-\alpha) \left(\frac{X_{1t}}{X_{2t}} \right)^{1-\theta},$$

$$X_{1t} = (1+\mu) MC_t Y_t + \alpha\beta E_t \left[\left(\frac{Y_{t+1}}{Y_t} \right)^{-\sigma} \Pi_{t+1}^{\theta} X_{1,t+1} \right],$$

$$X_{2t} = (1-\tau_t) Y_t + \alpha\beta E_t \left[\left(\frac{Y_{t+1}}{Y_t} \right)^{-\sigma} \Pi_{t+1}^{\theta-1} X_{2,t+1} \right],$$

$$\Delta_t = \alpha \Pi_t^{\theta} \Delta_{t-1} + (1-\alpha) \left(\frac{1-\alpha\Pi_t^{\theta-1}}{1-\alpha} \right)^{\frac{\theta}{\theta-1}}.$$

(i) Explain the role of each of the six equations above.

(ii) Which variable is purely cross-sectional?

(iii) Why is the nonlinear system not easily solvable in closed form?

5. Efficient steady state

Assume the benchmark steady state satisfies

$$\Pi = 1, \qquad \Delta = 1, \qquad MC = 1.$$

(i) Show that with zero inflation

$$1 = \alpha + (1-\alpha)\left(\frac{X_1}{X_2}\right)^{1-\theta} \quad \Rightarrow \quad X_1 = X_2.$$

(ii) In steady state, show that

$$X_1 = \frac{(1+\mu)MC\,Y}{1-\alpha\beta}, \qquad X_2 = \frac{(1-\tau)Y}{1-\alpha\beta}.$$

(iii) Deduce that

$$(1+\mu)MC = 1-\tau.$$

(iv) Show that choosing

$$\tau = -\mu$$

removes the monopoly distortion in steady state.

(v) Using results from section 3, derive

$$Y^* = A^{\frac{1+\varphi}{\sigma+\varphi}}.$$

(vi) Explain why this is the efficient level of output in the no-capital economy.

6. Log-linearised system

Log-linearise around the efficient steady state.

(i) Explain why, to first order,

$$\widehat{\Delta}_t = 0.$$

New Keynesian Phillips Curve

Start from

$$1 = \alpha\Pi_t^{\theta-1} + (1-\alpha)\left(\frac{X_{1t}}{X_{2t}}\right)^{1-\theta}.$$

(i) Show that log-linearisation yields

$$\pi_t = \frac{(1-\alpha)(1-\alpha\beta)}{\alpha}\left(mc_t + \widehat{\tau}_t\right) + \beta E_t\pi_{t+1}.$$

(ii) Starting from the labor-cost relation, show that

$$(1+\varphi)a_t + mc_t = (\sigma+\varphi)y_t.$$

(iii) At the efficient allocation, show that

$$y_t^* = \frac{1+\varphi}{\sigma+\varphi}a_t.$$

(iv) Deduce that

$$mc_t = (\sigma + \varphi)(y_t - y_t^*).$$

(v) Define the output gap

$$x_t \equiv y_t - y_t^*.$$

Show that the Phillips curve becomes

$$\pi_t = \frac{(1-\alpha)(1-\alpha\beta)(\sigma+\varphi)}{\alpha} x_t + \beta E_t \pi_{t+1} + \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} \widehat{\tau}_t.$$

(vi) Define

$$\kappa \equiv \frac{(1-\alpha)(1-\alpha\beta)(\sigma+\varphi)}{\alpha}, \qquad u_t \equiv \frac{(1-\alpha)(1-\alpha\beta)}{\alpha} \widehat{\tau}_t,$$

and write the NKPC in the compact form

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1} + u_t.$$

Demand curve

Start from the Euler equation

$$Y_t^{-\sigma} = \beta E_t \left[Y_{t+1}^{-\sigma} \frac{1+i_t}{\Pi_{t+1}} \right].$$

(i) Show that, to first order,

$$y_t = E_t y_{t+1} - \sigma^{-1}(i_t - E_t \pi_{t+1}).$$

(ii) Rewrite the equation in output-gap form:

$$x_t = -\sigma^{-1}(i_t - E_t \pi_{t+1} - r_t^*) + E_t x_{t+1},$$

where

$$r_t^* \equiv \sigma(E_t y_{t+1}^* - y_t^*).$$

(iii) Define the real interest rate

$$r_t \equiv i_t - E_t \pi_{t+1},$$

rewrite the IS relation as

$$x_t = -\sigma^{-1}(r_t - r_t^*) + E_t x_{t+1}.$$

and explain the economic meaning of r_t^*.

(iv) Rewrite the expression for r_t^* as a function of a_t, assuming $a_t = \rho_a a_{t-1} + \zeta_t$, where ζ is standard normal white noise.

Monetary policy rule

Assume

$$i_t = \phi_\pi \pi_t + \phi_x x_t + \varepsilon_t.$$

(i) Explain the role of ϕ_π and ϕ_x.

(ii) What does ε_t capture?

7. The three-equation New Keynesian model

Collect the three equations:

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1} + u_t,$$
$$x_t = -\sigma^{-1}(i_t - E_t \pi_{t+1} - r_t^*) + E_t x_{t+1},$$
$$i_t = \phi_\pi \pi_t + \phi_x x_t + \varepsilon_t.$$

And define the shocks:

$$u_t = \rho_u u_{t-1} + \varepsilon_t^u,$$
$$a_t = \rho_a a_{t-1} + \varepsilon_t^a,$$
$$\varepsilon_t = \rho_\varepsilon \varepsilon_{t-1} + \varepsilon_t^m.$$

(i) For each equation, state clearly whether it is an aggregate supply relation, aggregate demand relation, or policy rule.

(ii) Explain in one sentence what each equation is pinning down.

9. Equilibrium determinacy and the Taylor principle

Substitute the Taylor rule into the IS curve:

$$x_t = -\sigma^{-1}\left(\phi_\pi \pi_t + \phi_x x_t + \varepsilon_t - E_t \pi_{t+1} - r_t^*\right) + E_t x_{t+1}.$$

Together with

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1} + u_t,$$

this gives a forward-looking 2×2 system.

(i) Show that the system can be written as

$$A_0 \begin{pmatrix} E_t x_{t+1} \\ E_t \pi_{t+1} \end{pmatrix} = A_1 \begin{pmatrix} x_t \\ \pi_t \end{pmatrix} + B v_t,$$

with

$$A_0 = \begin{pmatrix} \sigma & 1 \\ 0 & \beta \end{pmatrix}, \qquad A_1 = \begin{pmatrix} \sigma + \phi_x & \phi_\pi \\ \kappa & 1 \end{pmatrix}.$$

(ii) Show that equivalently

$$E_t z_{t+1} = A z_t + \widetilde{B} v_t, \qquad A = A_0^{-1} A_1.$$

(iii) Explain what determinacy means in this model.

(iv) State the Blanchard-Kahn condition for this system.

(v) Explain why, in the purely forward-looking 2×2 setup, determinacy requires both eigenvalues of A to lie outside the unit circle.

(vi) Show that the generalized Taylor principle is

$$\phi_\pi > 1 - \frac{(1-\beta)\phi_x}{\kappa}.$$

(vii) Show that when $\phi_x = 0$, this reduces to

$$\phi_\pi > 1.$$

(viii) Explain the economic intuition behind the Taylor principle.

(ix) Explain why, if $\phi_\pi < 1$, inflation expectations may become self-fulfilling.

10. Solving the model with the method of undetermined coefficients

Consider the baseline system

$$\pi_t = \kappa x_t + \beta E_t \pi_{t+1} + u_t,$$
$$x_t = -\sigma^{-1}(i_t - E_t \pi_{t+1} - r_t^*) + E_t x_{t+1},$$
$$i_t = \phi_\pi \pi_t + \phi_x x_t + \varepsilon_t,$$

and define the real interest rate

$$r_t \equiv i_t - E_t \pi_{t+1}.$$

In each case below, shut down the other shocks and solve one shock at a time.

Cost-push shock

Assume

$$u_t = \rho_u u_{t-1} + \varepsilon_t^u, \qquad r_t^* = 0, \qquad \varepsilon_t = 0.$$

Conjecture

$$x_t = a_u u_t, \qquad \pi_t = b_u u_t.$$

(i) Show that

$$E_t x_{t+1} = a_u \rho_u u_t, \qquad E_t \pi_{t+1} = b_u \rho_u u_t.$$

(ii) Substituting into the Phillips curve and the IS curve, show that

$$b_u = \kappa a_u + \beta \rho_u b_u + 1,$$
$$\left[\sigma(1-\rho_u) + \phi_x\right] a_u + (\phi_\pi - \rho_u) b_u = 0.$$

(iii) Show that, defining

$$D_u \equiv \left[\sigma(1-\rho_u) + \phi_x\right](1 - \beta\rho_u) + \kappa(\phi_\pi - \rho_u),$$

the undetermined coefficients are

$$a_u = -\frac{\phi_\pi - \rho_u}{D_u}, \qquad b_u = \frac{\sigma(1-\rho_u) + \phi_x}{D_u}.$$

(iv) Show that the nominal interest rate satisfies

$$i_t = (\phi_\pi b_u + \phi_x a_u) u_t.$$

(v) Show that the real interest rate satisfies

$$r_t = (\phi_\pi b_u + \phi_x a_u - \rho_u b_u) u_t$$

and therefore

$$r_t = -\sigma(1-\rho_u) a_u \, u_t = \frac{\sigma(1-\rho_u)(\phi_\pi - \rho_u)}{D_u} u_t.$$

(vi) Show that $\phi_\pi > \rho_u$,

$$a_u < 0, \qquad b_u > 0, \qquad r_t > 0.$$

and interpret the results

(vii) Show that

$$i_t = \frac{\phi_\pi \sigma(1-\rho_u) + \phi_x \rho_u}{D_u} u_t,$$

what is the response of i_t to a positive cost-push shock?

Natural-rate shock

Assume

$$r_t^* = \rho_r r_{t-1}^* + \varepsilon_t^r, \qquad u_t = 0, \qquad \varepsilon_t = 0.$$

Conjecture

$$x_t = a_r r_t^*, \qquad \pi_t = b_r r_t^*.$$

(i) Show that

$$E_t x_{t+1} = a_r \rho_r r_t^*, \qquad E_t \pi_{t+1} = b_r \rho_r r_t^*.$$

(ii) Show that substitution into the two equilibrium conditions gives

$$b_r = \kappa a_r + \beta \rho_r b_r,$$

$$\big[\sigma(1-\rho_r) + \phi_x\big] a_r + (\phi_\pi - \rho_r) b_r = 1.$$

(iii) Show that, defining

$$D_r \equiv \big[\sigma(1-\rho_r) + \phi_x\big](1-\beta\rho_r) + \kappa(\phi_\pi - \rho_r),$$

the undetermined coefficients are

$$a_r = \frac{1-\beta\rho_r}{D_r}, \qquad b_r = \frac{\kappa}{D_r}.$$

(iv) Show that

$$i_t = (\phi_\pi b_r + \phi_x a_r) r_t^*.$$

(v) Show that

$$r_t = (\phi_\pi b_r + \phi_x a_r - \rho_r b_r) r_t^*$$

and therefore

$$r_t = [1 - \sigma(1-\rho_r) a_r]\, r_t^* = \frac{\phi_x(1-\beta\rho_r) + \kappa(\phi_\pi - \rho_r)}{D_r} r_t^*.$$

(vi) Show that, if $D_r > 0$ and $\phi_\pi > \rho_r$,

$$a_r > 0, \qquad b_r > 0, \qquad r_t > 0.$$

(vii) Show that

$$i_t = \frac{\phi_\pi \kappa + \phi_x(1-\beta\rho_r)}{D_r} r_t^*,$$

and deduce that, if $D_r > 0$, the impact response of i_t to a positive natural-rate shock is positive.

Monetary policy shock

Assume

$$\varepsilon_t = \rho_\varepsilon \varepsilon_{t-1} + \varepsilon_t^m, \qquad u_t = 0, \qquad r_t^* = 0.$$

Conjecture

$$x_t = a_\varepsilon \varepsilon_t, \qquad \pi_t = b_\varepsilon \varepsilon_t.$$

(i) Show that

$$E_t x_{t+1} = a_\varepsilon \rho_\varepsilon \varepsilon_t, \qquad E_t \pi_{t+1} = b_\varepsilon \rho_\varepsilon \varepsilon_t.$$

(ii) Show that substitution into the two equilibrium conditions gives

$$b_\varepsilon = \kappa a_\varepsilon + \beta \rho_\varepsilon b_\varepsilon,$$

$$\big[\sigma(1-\rho_\varepsilon) + \phi_x\big] a_\varepsilon + (\phi_\pi - \rho_\varepsilon) b_\varepsilon = -1.$$

(iii) Show that, defining

$$D_\varepsilon \equiv \big[\sigma(1-\rho_\varepsilon) + \phi_x\big](1-\beta\rho_\varepsilon) + \kappa(\phi_\pi - \rho_\varepsilon),$$

the undetermined coefficients are

$$a_\varepsilon = -\frac{1-\beta\rho_\varepsilon}{D_\varepsilon}, \qquad b_\varepsilon = -\frac{\kappa}{D_\varepsilon}.$$

(iv) Show that

$$i_t = (\phi_\pi b_\varepsilon + \phi_x a_\varepsilon + 1)\varepsilon_t.$$

(v) Show that

$$r_t = (\phi_\pi b_\varepsilon + \phi_x a_\varepsilon + 1 - \rho_\varepsilon b_\varepsilon)\varepsilon_t$$

and therefore

$$r_t = -\sigma(1-\rho_\varepsilon) a_\varepsilon \, \varepsilon_t = \frac{\sigma(1-\rho_\varepsilon)(1-\beta\rho_\varepsilon)}{D_\varepsilon}\varepsilon_t.$$

(vi) Show that, if $D_\varepsilon > 0$,

$$a_\varepsilon < 0, \qquad b_\varepsilon < 0, \qquad r_t > 0.$$

(vii) Show that

$$i_t = \frac{\sigma(1-\rho_\varepsilon)(1-\beta\rho_\varepsilon) - \kappa\rho_\varepsilon}{D_\varepsilon}\varepsilon_t.$$

Hence show that the sign of the nominal-rate response is positive provided

$$\sigma(1-\rho_\varepsilon)(1-\beta\rho_\varepsilon) > \kappa\rho_\varepsilon.$$